Differential forms and the Hölder equivalence problem

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Heis denotes *Heisenberg group*, the group of matrices of the form $\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}$ with real entries. *Dilatations* $\delta_t(x,y,z)=(tx,ty,t^2z)$ are group automorphisms. Its Lie algebra has basis

$$X=\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix},\quad Y=\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix},\quad Z=\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix},$$

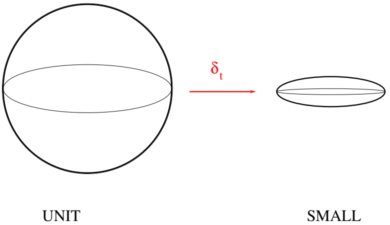
with $[X,Y]=Z$ and all other brackets vanish. So $\{X,Y\}$ is a bracket generating. Its trajectories (paths tangent to $V_1=\mathrm{span}(\{X,Y\})$) are called *horizontal paths*. The sub-Riemannian (or Carnot-Carathéodory) metric on *Heis*,

$$d(p,q)=\inf\{\mathrm{length}(\gamma)\,;\,\gamma \text{ horizontal path joining } p \text{ to } q\},$$

is left-invariant and multiplied by t by δ_t.

The distribution of planes V_1 is the kernel of the left-invariant differential 1-form $\theta=dz-x\,dy$.

Small R-balls are squeezed in the z direction, they have volume R^4,



Hausdorff dimension equals 4. Vertical lines have Hausdorff dimension 2. Only (rectifiable) horizontal curves have finite 1-dimensional Hausdorff measure.

Metricly very different from Euclidean 3-space. How much ?

Lecture 1

Metric problems in sub-Riemannian geometry

What do sub-Riemannian manifolds say, from the metric viewpoint ? Here is a choice of problems which are open even in dimension 3.

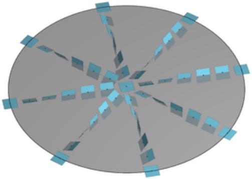
1. Bi-Lipschitz equivalence of contact manifolds
2. Bi-Lipschitz embedding *Heis* in L^1
3. Bi-Lipschitz embedding snowflakes of *Heis* in $\mathbb{R}^N$
4. Hölder equivalence problem
5. Hölder-Lipschitz equivalence problem
6. Quasi-symmetric Hölder-Lipschitz equivalence problem

I comment on each of them. The sequel of the course will focus on 4.

A *contact structure* on a 3-manifold is a step 2 plane distribution.

Theorem (Bennequin 1983, Eliashberg 1989, 1992)

Up to C^1 diffeomorphisms, there are exactly 2 different contact structures on $\mathbb{R}^3$.



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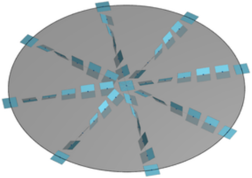

Question

Does there exist a locally bi-Lipschitz homeomorphism between the tight (standard) contact structure and the overtwisted (Lutz) contact structure ?

At first sight, none of the classical contact invariants (overtwisted disks, Lutz tubes, Thurston-Bennequin numbers,...) seems to be preserved by bi-Lipschitz homeomorphism.

Theorem (Semmes 1996, Pauls 2001)

No bi-Lipschitz embedding of Heis in L^p if $1 < p < \infty$.

Proof A.e., there is a differential, a group homomorphism $Df : Heis \to L^p$, which is bi-Lipschitz. But Df has a kernel, contradiction. q.e.d.

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Theoretical computer science (hardness of approximation of SPARSEST CUT) motivates the

Conjecture (Lee-Naor 2006)

For every 1-Lipschitz map of the R-ball in Heis into L^1, there are points whose distance is decreased at least by a factor of $(\log R)^{1/2-\epsilon}$, $\forall \epsilon > 0$.

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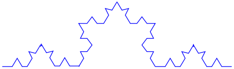
Known *for L^1 with some weaker exponent* $\delta < \frac{1}{2}$ (Cheeger-Kleiner-Naor 2010). Quantitative differentiability.
Known *if L^1 is replaced by L^p, $1 < p \leq 2$, with sharp exponent* $\frac{1}{2}$ (Lafforgue-Naor 2012). Functional inequality.

Embedding theory raises new questions: $Heis^m$ is Markov 4-convex (Li, 2014).

Remark: a log perturbation of the Carnot metric bi-Lipschitz embeds in L^2. Can one do this in finite dimension ?



A *snowflake* of metric space (X, d_X) is $X^{1-\epsilon} = (X, d_X^{1-\epsilon})$.
The *Assouad-Bouligand dimension* $\dim_{AB}(X)$ is the infimal δ such that the number of disjoint r-balls in an R-ball is $O((\frac{R}{r})^\delta)$.

Theorem (Assouad 1983)

If $\dim_{AB}(X) \leq \delta$, $X^{1-\epsilon}$ *has a L-bi-Lipschitz embedding into* $\mathbb{R}^N$, *with* $N = N(\delta, \epsilon)$, $L = L(\delta, \epsilon)$.

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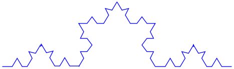

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Question

What is the optimal embedding dimension for Heis ?

Easy: $N \geq 5$.

Remark: The obvious map $\mathbb{R}^3 \to$ *Heis* is $C^{1/2}$-Hölder continuous, and its inverse is Lipschitz.

Question (Hölder equivalence problem, Gromov 1993)

Let M be a sub-Riemannian manifold. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism from Euclidean space to M which is C^α-Hölder continuous ?

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Theorem (Gromov 1993)

Let metric space X have dimension n, Hausdorff dimension Q. Then $\alpha(X) \leq \frac{n}{Q}$.
Let sub-Riem. M have dimension n, Hausdorff dimension Q. Then $\alpha(M) \leq \frac{n-1}{Q-1}$.
Let M be a $2m+1$-dimensional contact manifold. Then $\alpha(M) \leq \frac{m+1}{m+2}$ $(\leq \frac{2m}{2m+1})$.

Definition

Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is bi-Lipschitz to a δ-pinched complete simply connected Riemannian manifold.*

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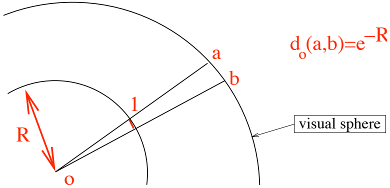
Facts.

- Negatively curved manifolds M have a visual sphere ∂M, equipped with a visual metric.
- If M is δ-pinched, polar coordinates define a $C^{\sqrt{-\delta}}$-Hölder homeomorphism from the round sphere $S \to \partial M$, with 1-Lipschitz inverse.
- Bi-Lipschitz maps between negatively curved Riemannian manifolds induce *quasisymmetric* maps between ideal boundaries.



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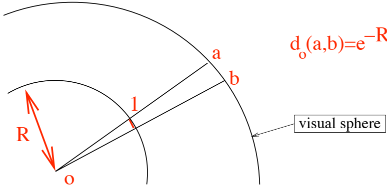

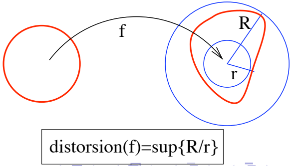

Definition

$\alpha_{qs}(X) = \sup\{\alpha \in (0,1) \,|\, \exists$ *locally a* C^α *homeomorphism with Lipschitz inverse from Euclidean space to a metric space quasisymmetric to* $X\}$

By definition, $\sqrt{-\delta(M)} \leq \alpha_{qs}(\partial M)$.

Example: The visual boundary of complex hyperbolic plane is a sub-Riemannian 3-sphere, quasisymmetric to *Heis*. Note that $\alpha_{qs}(Heis) \geq \alpha(Heis) \geq \frac{1}{2}$.

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Conjecture

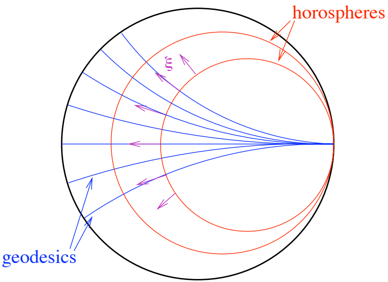
$\alpha_{qs}(Heis) = \frac{1}{2}$.

If the proof survives and time permits, I will explain

Theorem

$\alpha_{qs}(Heis) \leq \frac{2}{3}$.

More on complex hyperbolic spaces. The ball $B \subset \mathbb{C}^m$ has biholomorphism group $PU(m,1)$, which preserves a complete Riemannian metric on B and the contact structure of complex tangents on ∂B. $PU(m,1)$ has a subgroup $S = \mathbb{R} \rtimes Heis^{2m-1}$ acting simply transitively on the ball. The conjugation action of $\mathbb{R}$ on $Heis^{2m-1}$ is by dilations. The induced metric on S is of the form $dt^2 + \delta_t^* g_0$, for some left-invariant Riemannian metric g_0 on $Heis^{2m-1}$. It is $-\frac{1}{4}$-pinched in the t direction, since $\delta_t^* g_0 = e^{2t} g_{V_1} + e^{4t} g_{V_2}$, thus in all directions by $PU(m,1)$-symmetry. $\mathbb{R}$ factors are geodesics, $Heis^{2m-1}$-orbits are horospheres.



The visual sphere identifies with ∂B equipped with a sub-Riemannian metric.

Lecture 2

Gromov's dimension approach to the Hölder equivalence problem

Source: Gromov's *Carnot-Carathéodory spaces seen from within*.

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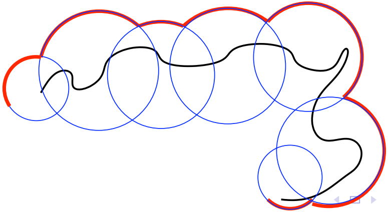
It follows that the boundary of any non smooth domain Ω has Hausdorff dimension at least $Q-1$. Indeed, cover $\partial\Omega$ with balls B_j and apply (*) to $\Omega \cup \bigcup B_j$. This gives a lower bound on $\mathcal{H}^{Q-1}(\partial(\bigcup B_j)) \leq \sum \mathcal{H}^{Q-1}(\partial B_j) \leq const.\, \sum \mathrm{diameter}(B_j)^{Q-1}$.



Proof of Euclidean isoperimetric inequality (with unsharp constant)

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Fix point p. Let $vol = dx_1 \wedge \dots \wedge dx_n$ be the volume form. Let $\xi(q) = \frac{q-p}{|q-p|^n}$ be a radial vectorfield and $\omega_p = \iota_\xi \omega$.

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$$\mathrm{vol}(D) = \int_D \Big(\int_{\partial D} \omega_p\Big)\, dp \leq \int_{D\times\partial D} |p-q|^{1-n}\, dq\, dp = \int_{\partial D} \Big(\int_D |p-q|^{n-1}\, dp\Big)\, dq.$$

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Key step is estimate $|\omega_p| \leq |p-q|^{1-n}$. It follows from homogeneity under dilations,

$$\delta_t^* \omega_0 = \omega_0,$$

since $n-1$-forms have pointwise homogeneity $n-1$.

On Carnot groups, $n-1$-forms come in several homogeneities called *weights*.

Definition

On a Carnot group, a left invariant form $\lambda \in \Lambda^k \mathfrak{g}^*$ *has weight* w *if* $\delta_t^* \lambda = t^w \lambda$. *A smooth differential form* ω *has weight* w *if it is a linear combination of left-invariant forms of weight* w.

Example: on *Heis*, dx, dy, $\theta = dz - xdy$ are a basis of invariant 1-forms, with dx, dy of weight 1 and θ of weight 2. $2xdx \wedge dy$ has weight 2, $(ydx + 2xdy) \wedge \theta$ has weight 3.

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Lemma

On a Carnot group G, *if a smooth differential form* ω *of weight* w *satisfies* $\delta_2^* \omega = \omega$ *on* $G \setminus \{e\}$, *then* $|\omega(q)| \leq \mathrm{const.}\, d(q,e)^{-w}$.

Indeed, $\{\delta_t^* \omega \,;\, t \in \mathbb{R}\}$ is a compact set, it is bounded on the unit sphere S. Let $\omega = \sum a_i \lambda_i$ with left-invariant λ_i. If $d(q,e) = t$, $q = \delta_t(q') \in S \subset G$,

$$d(q,e)^w |\omega|(q) = \sum |t^w a_i(q)| = |\sum a_i \circ \delta_t(q') \delta_t^* \lambda_i| = |\delta_t^* \omega|(q') \leq \mathrm{const.}$$

A closed $n-1$-form ω on $G \setminus \{e\}$ such that $\delta_2^*\omega_0 = \omega_0$, with integral 1 on spheres, is a representative of the relevant class in $H^{n-1}(M)$, where $M = (G \setminus \{e\})/ < \delta_2 >$.

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Every cohomology class in $H^{n-1}(M)$ has a representative of weight $Q-1$.

Proof for *Heis*. Let $\omega = a\, dx \wedge dy + \beta \wedge \theta$ be a closed 2-form. Then

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Proof in general. All left-invariant $n-1$-forms are closed. Under dilations, the cohomology of left-invariant forms splits

$$H^{n-1}(\mathfrak{g}) = \bigoplus^{w} H^{n-1,w}(\mathfrak{g}).$$

Poincaré duality gives $H^{n-1,w} \simeq H^{1,Q-w} = 0$ unless $Q - w = 1$. Therefore for every left-invariant form λ of weight $w \neq Q-1$, $\exists$ left-invariant μ of weight w such that $d\mu = \lambda$. Write $\omega = \sum a_i \lambda_i$, λ_i left-invariant of weight w_i. If $\lambda_i = d\mu_i$, substracting $d(a_i \mu_i)$ replaces $a_i \lambda_i$ by $da_i \wedge \mu_i$ of weight $w_i + 1$. After finitiely many steps, only terms of weight $Q-1$ remain. q.e.d.

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To get lower bounds on Hausdorff dimension of subsets, Gromov constructs local foliations by horizontal submanifolds. If there are enough such dimension k foliations, all subsets of topological dimension $n-k$ have Hausdorff dimension $\geq Q-k$, therefore $\alpha(M) \leq \frac{n-k}{Q-k}$.

Constructing horizontal submanifolds amounts to solving a system of PDE's. If $k=1$, it is an ODE, the method applies to all (equiregular) sub-Riemannian manifolds. Gromov solves the relevant PDE for contact $2m+1$-manifolds and $k=m$, and, more generally, for generic h-dimensional distributions, and k such that $h-k \geq (n-h)k$.

Lecture 3

Gromov's cochain approach to the Hölder equivalence problem

Source: Gromov's *Carnot-Carathéodory spaces seen from within*.

Definition

*On a metric space X, a (straight) q-*cochain of size ϵ *is a function c on $q+1$-uples of diameter $\leq \epsilon$. Its ϵ-absolute value is*

$$|c|_{\epsilon} = \sup\{c(\Delta)\,;\, diam(\Delta) \leq \epsilon\}.$$

In other words, straight cochains of size ϵ coincide with simplicial cochains on the simplicial complex whose vertices are points of X and a q-face joins $q+1$ vertices as soon as all pairwise distances are $\leq \epsilon$. Therefore, they form a complex $\mathcal{C}^{\cdot}_{\epsilon}$. There is a dual complex of chains $\mathcal{C}_{\cdot,\epsilon}$.

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Lemma

Assume X is a manifold with boundary, or bi-Hölder homeomorphic to such, then the inductive limit complex $\varinjlim \mathcal{C}^{\cdot}_{\epsilon}$ computes cohomology.

Definition

Given a cohomology class κ and a number $\nu > 0$, one can define the ν-norm

$$\|\kappa\|_{\nu} = \liminf_{\epsilon\to 0} \epsilon^{-\nu} \inf\{|c|_{\epsilon} \,|\, \textit{ cochains } c \textit{ of size } \epsilon \textit{ representing } \kappa\}.$$

Definition

Let X be a metric space, let $q \in \mathbb{N}$. Define the metric weight *$MW_q(X)$ as the supremum of numbers ν such that there exist arbitrarily small open sets $U \subset M$ and nonzero straight cohomology classes $\kappa \in H^q(U, \mathbb{R})$ with finite ν-norm $\|\kappa\|_\nu < +\infty$.*

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Proposition

In Euclidean space, all straight cocycles c representing a nonzero class κ of degree q satisfy $|c|_\epsilon \geq const.(\kappa)\,\epsilon^q$. In other words, $\|\kappa\|_q > 0$.

Proof . Fix a cycle c' such that $\kappa(c') > 0$. Subdivide it as follows : fill simplices with affine singular simplices, subdivide them and keep only their vertices. This does not change the homology class. The number of simplices of size ϵ thus generated is $\leq \mathrm{const.}(c')\,\epsilon^{-q}$. For any representative c of size ϵ of κ,

$$\kappa(c') = c(c') \leq \mathrm{const.}\,\epsilon^{-q}|c|_\epsilon.\ q.e.d.$$

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Corollary

Euclidean n-space has $MW_q \leq q$ for all $q = 1, \ldots, n-1$ (later, we shall see that $MW_q = q$).

Proposition

Let $f : X \to Y$ be a C^α-Hölder continuous homeomorphism. Let $\kappa \in H^q(Y, \mathbb{R})$. Then

$$\|\kappa\|_\nu < +\infty \Rightarrow \|f^*\kappa\|_{\nu\alpha} < +\infty.$$

In particular, $MW_q(X) \geq \alpha MW_q(Y)$. Consequence: $\alpha(M) \leq \frac{q}{MW_q(M)}$ for all q.

Proof . If σ is a straight simplex of size ϵ in X, $f(\sigma)$ has size $\epsilon' \leq \|f\|_{C^\alpha} \epsilon^\alpha$ in Y. If c is a representative of κ, f^*c is a representative of $f^*\kappa$, and

$$\begin{aligned} \epsilon'^{-\nu}|c|_{\epsilon'} &\geq \epsilon'^{-\nu}|c(f(\sigma))| \\ &= \epsilon'^{-\nu}|f^*c(\sigma)| \\ &\geq \|f\|_{C^\alpha}^{-\nu}\, \epsilon^{-\nu\alpha}|f^*c(\sigma)|. \end{aligned}$$

Therefore

$$\epsilon^{-\nu\alpha}|f^*c|_\epsilon \leq \|f\|_{C^\alpha}^{\nu}\, \epsilon'^{-\nu}|c|_{\epsilon'}.$$

This leads to

$$\|f^*\kappa\|_{\nu\alpha} \leq \|f\|_{C^\alpha}^{\nu}\, \|\kappa\|_\nu .\ q.e.d.$$

Let G be a Carnot group with Lie algebra $\mathfrak{g}$. Left-invariant differential forms on G split into homogeneous components under the dilations δ_ϵ,

$$\Lambda^{\cdot}\mathfrak{g}^* = \bigoplus_{w} \Lambda^{\cdot,w} \quad \text{where} \quad \Lambda^{\cdot,w} = \{\alpha \,|\, \delta_\epsilon^* \alpha = \epsilon^w \alpha\}.$$

Therefore Lie algebra cohomology splits $H^q(\mathfrak{g}) = \bigoplus_w H^{q,w}(\mathfrak{g})$.

Example

If $G = Heis^{2m+1}$ is the Heisenberg group, for each degree $q \neq 0, 2m+1$,

$$\Lambda^q \mathcal{G}^* = \Lambda^{q,q} \oplus \Lambda^{q,q+1},$$

where $\Lambda^{q,q} = \Lambda^q(V^1)^*$ and $\Lambda^{q,q+1} = \Lambda^{q-1}(V^1)^* \otimes (V^2)^*$.

Notation: $\Lambda^{q,\geq w} = \bigoplus_{w' \geq w} \Lambda^{q,w'}$. The space of differential forms which are smooth linear combinations of left-invariant forms from $\Lambda^{q,\geq w}$ is denoted by $\Omega^{q,\geq w}$.
Note that each $\Omega^{\cdot,\geq w}$ is a differential ideal in the algebra of all differential forms $\Omega^{\cdot}$.

Proposition

G Carnot group. Let $U \subset G$ be a bounded open set with smooth boundary. Let ω be a closed differential form on U of weight $\geq w$. Then, for every ϵ small enough, the cohomology class $\kappa \in H^q(U,\mathbb{R})$ of ω can be represented by a straight cocycle c_ϵ (maybe defined on a slightly smaller homotopy equivalent open set) such that $|c_\epsilon|_\epsilon \leq const.\,\epsilon^w$. In other words, $\|\kappa\|_w < +\infty$.

Proof Use exponential map to push affine simplices in the Lie algebra to the group. Fill in all straight simplices in G of unit size with such affine singular simplices σ_1. Apply δ_ϵ and obtain a filling σ_ϵ for each straight simplex σ in G of size ϵ. Define a straight cochain c_ϵ of size ϵ on U by

$$c_\epsilon(\sigma) = \int_{\sigma_\epsilon} \omega.$$

Since ω is closed, Stokes theorem shows that c_ϵ is a cocycle. Its cohomology class in $H^q(U',\mathbb{R}) \simeq H^q(U,\mathbb{R})$ is the same as ω's. By compactess, if $\lambda \in \Lambda^{\cdot,w}$, $\sup_{\sigma_1} \int_{\sigma_1} \lambda \leq \mathrm{const.}(\lambda)$, so $\sup_{\sigma_\epsilon} \int_{\sigma_\epsilon} \lambda \leq \mathrm{const.}(\lambda)\,\epsilon^w$. Summing over components of ω,

$$|c_\epsilon(\sigma)| \leq \mathrm{const.}(\omega)\,\epsilon^w. \ q.e.d.$$

Definition

G Carnot group. The algebraic weight $AW_q(G)$ as the largest w such that there exists arbitrarily small open sets with smooth boundary $U \subset G$ and nonzero classes in $H^q(U, \mathbb{R})$ which can be represented by closed differential forms of weight $\geq w$.

We just proved that $MW_q \geq AW_q$.

Corollary

Let G be a Carnot group. Then for all $q = 1, \dots, n-1$, $\alpha(G) \leq \frac{q}{AW_q}$.

So our goal now is to show that for certain Carnot groups, for certain degrees q, in every open set, every closed differential q-form is cohomologous to a form of high weight.

It turns out that the obstruction for cohomologing q-forms towards weight $> w$ is $H^{q,w}(\mathfrak{g})$.

Theorem (Rumin 2005)

Let G be a Carnot group. Assume that, in the cohomology of the Lie algebra, $H^{q,w'}(\mathfrak{g}) = 0$ for all $w' < w$. Then $AW_q(G) \geq w$.

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This will be proven later. First reformulate and illustrate. On Carnot groups, the grading of cohomology is compatible with Poincaré duality, $H^{q,w}(\mathfrak{g}) = H^{n-q,Q-w}(\mathfrak{g})$. So

$$H^{n-q}(\mathfrak{g}) = H^{n-q,\leq Q-w}(\mathfrak{g}) \quad \Rightarrow \quad AW_q(G) \geq w.$$

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Example

Degree $n-1$. On any Carnot Lie algebra $\mathfrak{g}$, closed 1-forms belong to $(V^1)^* = \Lambda^{1,1}$, so $H^1(\mathfrak{g}) = H^{1,1}(\mathfrak{g})$, and $AW_{n-1}(M) \geq Q-1$.

From now on, the source is Rumin's lecture notes *An introduction to spectral and differential geometry in Carnot-Carathéodory spaces.*

Example

$2m+1$-dimensional contact manifolds. Closed m-forms belong to $\Lambda^{m,m}$. Therefore $H^m(\mathfrak{g}) = H^{m,m}(\mathfrak{g})$, $AW_{m+1}(M) \geq m+2$ and $\alpha \leq \frac{m+1}{m+2}$.

Indeed, if $\omega \in \Lambda^{m,m+1}$, $\omega = \theta \wedge \phi$ where $\theta \in (V^2)^*$, $\phi \in \Lambda^{m-1,m-1}$, $(d\omega)^{m+1,m+1} = (d\theta) \wedge \phi \neq 0$ since $d\theta$ is symplectic on Δ.

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Example

Engel case. The nonzero weight spaces in $H^{\cdot}(\mathfrak{g})$ are $H^{0,0}$, $H^{1,1}$, $H^{2,3}$, $H^{2,4}$, $H^{3,6}$ and $H^{4,7}$. So the best bound on $\alpha(G)$ is $\frac{1}{2}$, achieved for degree 3. Disappointing.

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Quaternionic Heisenberg group $Heis_{\mathbb{H}}^{4m-1}$. Julg 1995 shows that $H^q(\mathfrak{g}) = H^{q,\geq q+2}$ if $q \geq 2m$ and $H^q(\mathfrak{g}) = H^{q,\geq q+3}$ if $q \geq 3m$. Thus $\alpha(G) \leq \frac{2m}{2m+2} = \frac{3m}{3m+3}$, obtained when considering degrees $2m$ and $3m$.

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The method just exposed seems to cover all presently known results on the Hölder homeomorphism problem.

Proof of Rumin's theorem in degree $n-1$: reformulation of argument in proof of isoperimetric inequality.

Notation: $d_0 =$ exterior differential on left-invariant forms, seen as a 0-order differential operator on differential forms.

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Write $\omega = \omega_{<Q-1} + \omega_{Q-1}$. $d_0 = 0$ in degree $n-1$ and $H^{n-1}(\mathfrak{g}) = H^{n-1,Q-1}$ imply that $\omega_{<Q-1} \in \mathrm{im}(d_0)$. Pick a linear inverse d_0^{-1} on $\mathrm{im}(d_0)$. Consider $r\omega = \omega - dd_0^{-1}(\omega_{<Q-1})$. Then weight$(r\omega) >$ weight(ω), unless $\omega_{<Q-1} = 0$. Therefore iterating leads to a cohomologous form of weight $Q-1$.

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General case: not all left-invariant q-forms are closed. Instead of considering $1 - dd_0^{-1}$ on closed forms, construct a homotopy of chain complexes $r = 1 - dd_0^{-1} - d_0^{-1}d$.

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General case: not all left-invariant q-forms are closed. Instead of considering $1 - dd_0^{-1}$ on closed forms, construct a homotopy of chain complexes $r = 1 - dd_0^{-1} - d_0^{-1}d$. Pick homogeneous complements W_0 of $\ker(d_0)$ in $\Lambda^{\cdot}\mathfrak{g}^*$, and E_0 of $\mathrm{im}(d_0)$ in $\ker(d_0)$.

$$\Lambda^{\cdot}\mathfrak{g}^* = \mathrm{im}(d_0) \oplus E_0 \oplus W_0.$$

Set $d_0^{-1} = 0$ on $E_0 \oplus W_0$ and extend to $\Lambda^{\cdot}\mathfrak{g}^*$ using the inverse of $d_0 : \mathrm{im}(d_0) \to W$. Denote by $\pi_0 : \Lambda^{\cdot}\mathfrak{g}^* \to E_0$ the projector. d_0^{-1} and π_0 extend to differential forms, denote by $\mathcal{E}_0 = \mathrm{im}(\pi_0)$.

Theorem (Rumin 1999)

Let $r = 1 - dd_0^{-1} - d_0^{-1}d$. Iterates of r stabilise to a differential operator π which is again a homotopy equivalence of chain complexes. π is a projector onto the subcomplex

$$\mathcal{E} = \ker(d_0^{-1}) \cap \ker(d_0^{-1}d).$$

The restriction of π_0 to $\mathcal{E}$ is an isomorphism onto $\mathcal{E}_0$, with inverse the restriction of π to $\mathcal{E}_0$.

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Corollary: estimate of algebraic weight follows. If ω is closed, $\pi(\omega) = \pi \circ \pi_0 \circ \pi(\omega)$ is cohomologous to it. The weights present is $E_0 = \mathrm{im}(\pi_0)$ are those of the cohomology. π, like d, does not decrease weights. Thus weight$(\pi(\omega))$ is as high as the minimum weight in cohomology.

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Example: *Heis*.
$\ker(d_0) = \mathrm{span}\{1, dx, dy, dx \wedge dy, dx \wedge \theta, dy \wedge \theta, dx \wedge dy \wedge \theta\}$.
$\mathrm{im}(d_0) = \mathrm{span}\{dx \wedge dy\}$.
$W_0 = \mathrm{span}\{\theta\}$.
$E_0 = \mathrm{span}\{1, dx, dy, dx \wedge \theta, dy \wedge \theta, dx \wedge dy \wedge \theta\}$.
No need to iterate. $\pi = r$ maps $adx + bdy + c\theta$ to $adx + bdy + (Ya - Xb)\theta$ and $edx \wedge dy + fdx \wedge \theta + gdy \wedge \theta$ to $(Xe + f)dx \wedge \theta + (Ye + g)dy \wedge \theta$.

Proof 1. *Stabilization.* By construction, $r = 1$ on $\mathcal{E}$. Let $\mathcal{W}$ denote the space of differential forms which belong to W at each point. Then, on W, r is nilpotent. Indeed, $d_0^{-1} = 0$ on W so $r = 1 - d_0^{-1}d = -d_0^{-1}(d - d_0)$ maps $\mathcal{W}$ to itself and strictly increases weight. Since $rd = dr$, the same is true on $d\mathcal{W}$ and thus on $\mathcal{F} := \mathcal{W} + d\mathcal{W}$.

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2. *Claim*: $\Omega^\cdot = \mathcal{E} \oplus \mathcal{F}$.
Since $N = d_0^{-1}(d - d_0)_{|\mathcal{W}}$ is nilpotent, $1 + N$ has a differential inverse $P = 1 + \sum(-1)^i N^i$ defined on $\mathcal{W}$. Set $Q = Pd_0^{-1} : \Omega^\cdot \to \mathcal{W}$. We check that $\pi = 1 - Qd - dQ$ is the projector onto $\mathcal{E}$ with kernel $\mathcal{F}$. By definition, $d_0^{-1}(\mathcal{E}) = 0$, $d_0^{-1}d(\mathcal{E}) = 0$ so $\pi = 1$ on $\mathcal{E}$. Also $d_0^{-1}Q = 0$, $d_0^{-1}dQ = (1 + N)Pd_0^{-1} = d_0^{-1}$ so $d_0^{-1}\pi = 0$. Since $d\pi = \pi d$, $d_0^{-1}d\pi = 0$ so $\mathrm{im}(\pi) \subset \mathcal{E}$, thus π is a projector onto $\mathcal{E}$.
Use $\ker(\pi) = \mathrm{im}(dQ + Qd)$. $\mathrm{im}(Q) \subset \mathcal{W}$ so $\mathrm{im}(dQ) \subset \lceil\mathcal{W}$, and $\mathrm{im}(Qd + dQ) \subset \mathcal{W} + d\mathcal{W}$. Conversely, on $\mathcal{W}$, $Qd = Pd_0^{-1}d = 1$ so $\mathcal{W} \subset \ker(\pi)$. Since $d\pi = \pi d$, $d\mathcal{W} \subset \ker(\pi)$, so $\ker(\pi) = \mathcal{F}$.

Proof 1. *Stabilization*. By construction, $r=1$ on $\mathcal{E}$. Let $\mathcal{W}$ denote the space of differential forms which belong to W at each point. Then, on W, r is nilpotent. Indeed, $d_0^{-1}=0$ on W so $r=1-d_0^{-1}d=-d_0^{-1}(d-d_0)$ maps $\mathcal{W}$ to itself and strictly increases weight. Since $rd=dr$, the same is true on $d\mathcal{W}$ and thus on $\mathcal{F}:=\mathcal{W}+d\mathcal{W}$.

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Use $\ker(\pi)=\mathrm{im}(dQ+Qd)$. $\mathrm{im}(Q)\subset\mathcal{W}$ so $\mathrm{im}(dQ)\subset\lceil\mathcal{W}$, and $\mathrm{im}(Qd+dQ)\subset\mathcal{W}+d\mathcal{W}$. Conversely, on $\mathcal{W}$, $Qd=Pd_0^{-1}d=1$ so $\mathcal{W}\subset\ker(\pi)$. Since $d\pi=\pi d$, $d\mathcal{W}\subset\ker(\pi)$, so $\ker(\pi)=\mathcal{F}$.

3. *π and π_0 are inverses of each other on $\mathcal{E}_0$ (resp. $\mathcal{E}$).* Since $\mathrm{im}(\pi_0)=\mathcal{E}_0\subset\ker(d_0^{-1})\subset\ker(Q)$, $Q\pi_0=0$. Since $\mathrm{im}(Q)\subset\mathcal{W}\subset\ker(\pi_0)$, $\pi_0 Q=0$. Thus $\pi_0\circ\pi\circ\pi_0=\pi_0(1-Qd-dQ)\pi_0=\pi_0$, i.e. $\pi_0\circ\pi_{|\mathcal{E}_0}=1$. Since $\mathcal{E}\subset\ker(d_0^{-1})=\mathcal{W}\oplus\mathcal{E}_0$ and $\mathcal{W}\subset\ker(\pi_0)$, $\ker(\pi_{0|\mathcal{E}})=\mathcal{W}\cap\mathcal{E}=\{0\}$, so $\pi_{0|\mathcal{E}}$ is injective, and $\pi\circ\pi_{0|\mathcal{E}}=1$.

Definition

The differential operator $d_c = \pi_0 d\pi$ on $\mathcal{E}_0$ is Rumin's complex. Computes cohomology.

Example: *Heis*. $\mathcal{E}_0^0 = \Omega^0$, $\mathcal{E}_0^3 = \Omega^3 0$, $\mathcal{E}_0^1 = \mathrm{span}\{dx, dy\}$, $\mathcal{E}_0^2 = \mathrm{span}\{dx \wedge \theta, dy \wedge \theta\}$.
On forms of degrees 0, 2 and 3, $d_c = d$. On 1-forms, d_c is a second order operator, $d_c(adx + bdy) = (YXa - XXb)dx \wedge \theta + (-XYb + YYa)dy \wedge \theta$.

Definition

The differential operator $d_c = \pi_0 d\pi$ on $\mathcal{E}_0$ is Rumin's complex. Computes cohomology.

Example: *Heis*. $\mathcal{E}_0^0 = \Omega^0$, $\mathcal{E}_0^3 = \Omega^3 0$, $\mathcal{E}_0^1 = \mathrm{span}\{dx, dy\}$, $\mathcal{E}_0^2 = \mathrm{span}\{dx \wedge \theta, dy \wedge \theta\}$. On forms of degrees 0, 2 and 3, $d_c = d$. On 1-forms, d_c is a second order operator, $d_c(adx + bdy) = (YXa - XXb)dx \wedge \theta + (-XYb + YYa)dy \wedge \theta$.

Applications

Rumin 1994, 2000, Ge 1994. Asymptotics for the spectrum of the Laplace-Beltrami operator on differential forms for sequences of Riemannian metrics converging to a sub-Riemannian metric.

Julg-Kasparov 1995. Baum-Connes conjecture for $SU(n, 1)$.

Rumin 2002. Estimates on Novikov-Shubin invariants of nilpotent groups (large scale spectral invariants).

Julg 2002. Baum-Connes conjecture for $Sp(n, 1)$.

Biquard-Herzlich-Rumin 2006. Expressing secondary invariants of compact CR-manifolds.

Lecture 4

Quasisymmetric Hölder equivalence problem

Recall Gromov's slogan:

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$\exists$ k-dimensional subset with Hausdorff dimension $\leq k$	$\forall$ k-dimensional subset, Hausdorff dimension $\geq k'$

then $\alpha(Carnot) \leq \frac{k}{k'}$.

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Furthermore, the argument uses quasisymmetric invariants, whence

$$\alpha_{qs}(Carnot) \leq \frac{k}{k'}.$$

Let $f:\mathbb{R}^n \to G$ be a Hölder homeomorphism with Lipschitz $f^{-1}:G\to\mathbb{R}^n$. Let $v:\mathbb{R}^n\to\mathbb{R}$ be a coordinate function. Then $u=v\circ f^{-1}$ is Lipschitz. Imagine that *coarea inequality* holds:

$$\int_G Lip_u^Q \leq \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} Lip_u^{Q-1} \right) dt \leq \mathrm{const.} \int_{\mathbb{R}} \mathcal{H}^{Q-1}(u^{-1}(t))\, dt. \qquad (1)$$

Here, Lip_u denotes the local Lipschitz constant. Since, for non constant u, $\int_X Lip_u^Q > 0$, this shows that there exists $t\in\mathbb{R}$ such that $\mathcal{H}^{Q-1}(u^{-1}(t))>0$, and therefore $u^{-1}(t)$ has Hausdorff dimension at least $Q-1$.

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Strategy: replace conformally invariant integrals $\int Lip_u^Q$ with *packing measures* which are quasisymmetric invariants and satisfy coarea inequality in the right direction. If possible, use vector valued function u.

Let N be an integer, let $\ell \geq 1$. Let X be a metric space. An (N,ℓ)-*packing* is a countable collection of balls $\{B_j\}$ such that the collection of concentric balls ℓB_j has multiplicity $< N$.

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Let ϕ be a positive function on the set of balls in X. Define

$$\Phi^{p;\epsilon}_{N,\ell}(A) = \sup\{\sum_i \phi(B_i)^p \,;\, \{B_i\}\,(N,\ell)-\text{packing of } X,\ \text{centered on } A,\ \text{of mesh } \leq \epsilon\}.$$

Define the *packing pre-measure associated to* ϕ by

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Example: $\pi(B) =$radius(B) leads to "usual" packing measure Π^p and packing dimension. It is Hölder covariant.

Example: let $u : X \to M$ measure space. $e_u(B) =$measure(B) leads to *p-energy* E^p_u. It is quasisymmetry invariant: if g is quasisymmetric, $\forall\ell$, $\exists\ell'$ such that

$$E^p_{u,N,\ell}(A) \leq E^p_{u',N,\ell'}(g(A)).$$

Proposition

Let X be a metric space. Let $u : X \to M$ be a map to a measure space (M, μ). Then

$$E_u^p(X) \leq \int_M E_u^{p-1}(u^{-1}(m))\, d\mu(m).$$

$$\begin{aligned}
\sum_i \mu(u(B_i))^p &= \sum_i \left(\int_M 1_{u(B_i)}(m)\, d\mu(m)\right)\mu(u(B_i))^{p-1} \\
&= \int_M \left(\sum_i 1_{u(B_i)}(m)\mu(u(B_i))^{p-1}\right) d\mu(m) \\
&= \int_M \left(\sum_{\{i\,;\, m\in u(B_i)\}} \mu(u(B_i))^{p-1}\right) d\mu(m) \\
&\leq \int_M E_u^{p-1;2\epsilon}\, d\mu(m).\ q.e.d.
\end{aligned}$$

If u is Lipschitz and $\mu(B) \sim \mathrm{radius}(B)^d$ in M, $E_u^{p-1} \leq \Pi^{d(p-1)}$, thus

$$E_u^p > 0 \quad \Rightarrow \quad \exists m,\ \dim(u^{-1}(m)) \geq d(p-1).$$

Corollary

Let $f : \mathbb{R}^n \to G$ *be a* C^α*-Hölder homeomorphism, with Lipschitz*f^{-1}*. If* $v : \mathbb{R}^n \to \mathbb{R}^d$ *is a submersion and* $u = v \circ f^{-1}$ *satisfies* $E_u^{Q/d} > 0$*, then* $\alpha_{qs}(G) \leq \dfrac{n-d}{Q-d}$.

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Proposition

If $u:G\to\mathbb{R}$ is continuous and non constant, $E_u^Q>0$.

Corollary: $\alpha_{qs}(G)\leq\frac{n-1}{Q-1}$.

Proof Length-area method. Let Γ denote the family of unit segments parallel to a side of unit n-cube. For all positive functions ρ on the square,

$$\int_0^1\left(\int_\gamma \rho\,ds\right)d\gamma\leq\int\rho^n\,dx_1\ldots dx_n.$$

Replace integrals with packing (resp. covering) measures, apply to family of parallel horizontal line segments in G.

Proposition

Let X be a metric space. Let Γ be a family of subsets of X, equipped with a measure $d\gamma$. For each $\gamma\in\Gamma$, a probability measure m_γ is given on γ. Let $p\geq 1$. Assume that

$$\int_{\{\gamma\in\Gamma\,;\,\gamma\cap B\neq\emptyset\}} m_\gamma(\gamma\cap\ell B)^{1-p}\,d\gamma\leq\tau.$$

Then, for every function ϕ on the set of balls of X,

$$\Phi^p(X)\geq\frac{1}{\tau}\int_\Gamma\tilde\Phi^1(\gamma)^p\,d\gamma.$$

Proof Let $1_i(\gamma)=1$ iff $\gamma\cap B_i\neq\emptyset$. The balls such that $1_i(\gamma)=1$ cover γ, thus

$$\tilde\Phi^{1;\epsilon}(\gamma)\leq\sum_i\phi(B_i)1_i(\gamma)=\sum_i\phi(B_i)1_i(\gamma)m_\gamma(\gamma\cap\ell B_i)^{\frac{1-p}{p}}m_\gamma(\gamma\cap\ell B_i)^{\frac{p-1}{p}}.$$

Hölder's inequality gives

$$\tilde\Phi^{1;\epsilon}(\gamma)^p\quad\leq\quad\left(\sum_i\phi(B_i)^p1_i(\gamma)m_\gamma(\gamma\cap\ell B_i)^{1-p}\right)\left(\sum_i m_\gamma(\gamma\cap\ell B_i)\right)^{p-1}.$$

Integrate over Γ.

closed $n-1$-form	$\Leftrightarrow$	map to $\mathbb{R}^{n-1}$
closed $n-1$-form of weight $Q-1$	$\Leftrightarrow$	map to $\mathbb{R}^{n-1}$ with horizontal fibers

However, map to $\mathbb{R}^{n-1}$ is not Lipschitz in general.

On $Heis^3$, weight 3 2-forms give rise to cocycles such that $E^{4/3}<\infty$ thus $E^2=0$. But these correspond to maps which are not Lipschitz at all.

Question

Let X be a metric space which is quasisymmetric to an open subset of $Heis^3$. Let $u : X \to \mathbb{R}^2$ be Lipschitz and open. Show that $E_u^p > 0$ for all $p < 2$.

This would follow from

Question

Let $u : Heis \to \mathbb{R}^2$ be continuous and open. Assume that both components u^1 and u^2 satisfy $\tilde{E}^4_{u_i} < \infty$. Show that u is a.e. differentiable and infer that $E_u^2 > 0$.